Convex Network Flows

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MIT (Julia Lab) & Bain Capital

joint work with Guillermo Angeris and Alan Edelman

JuMP-dev 2024

Goals

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  - Natural way to specify the relevant optimization problem
  - Fast and parallelizable computational building blocks
- This talk: marriage of *convex optimization* and *network flows*

Outline

Motivation

Framework

Applications

Algorithm

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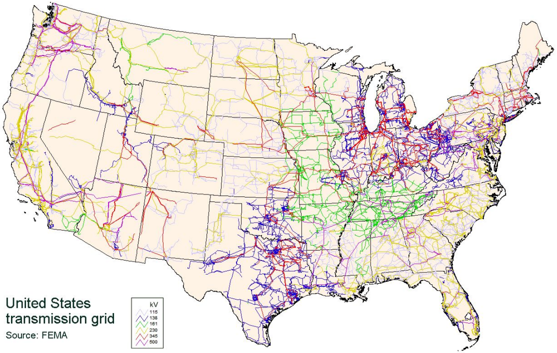
Nonlinear networks are everywhere!

- Power dispatch



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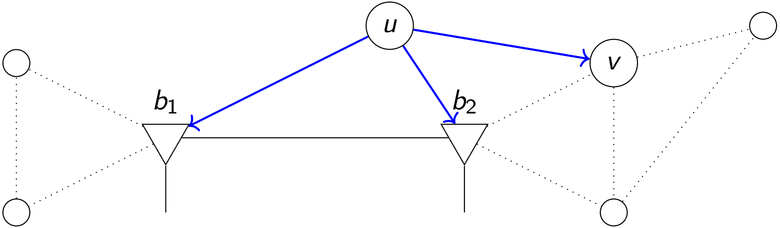

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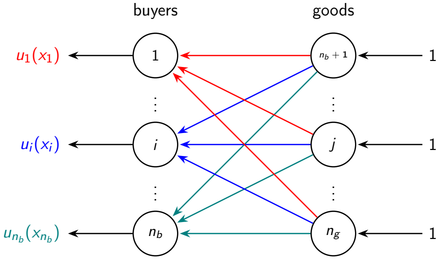

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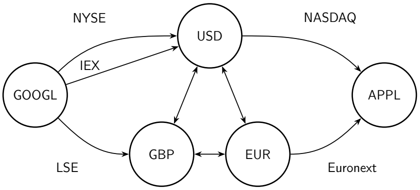
- Financial order routing



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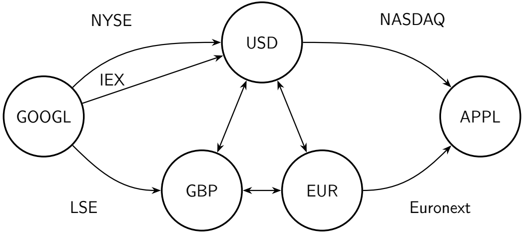
- ...

Motivating example: order routing in financial networks

(Diamandis et al., Financial Cryptography 2023)

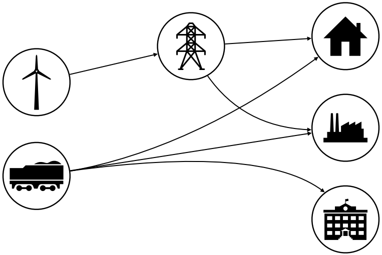
- **Goal:** convert fixed amount of Google stock to maximum amount of Apple stock
- Many venues on which you can place the order



- **Nonlinearity:** Price impact of trading: the more you trade, worse the price

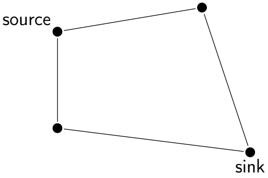
Motivating example: optimal power flow

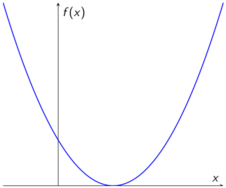
- **Goal:** generate power to meet demand at minimum cost
- Multiple sources and transmission lines to choose from



- **Nonlinearity:** Power loss: the more power transmitted, the more is dissipated

Perhaps a generalization?





Linear network flows

- Very fast to solve
- But not very expressive

$\longleftrightarrow$

Convex Optimization

- Very expressive
- But slow in general

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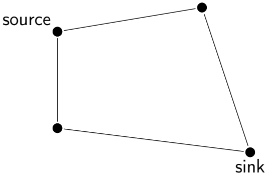

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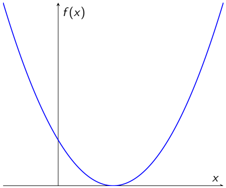

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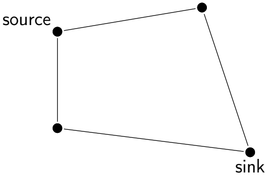

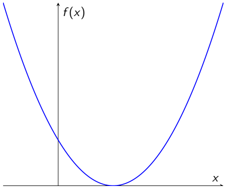

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A new framework:
Convex Network Flows

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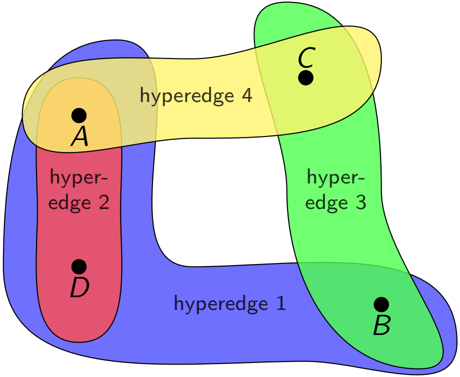
First generalization: hypergraphs

- Graph $\longrightarrow$ **hypergraph**: edges can connect more than 2 vertices



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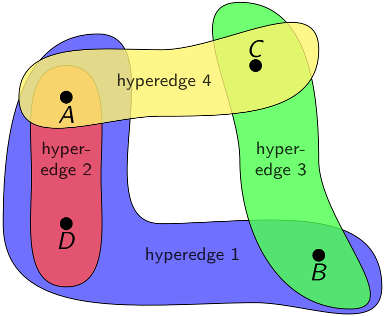
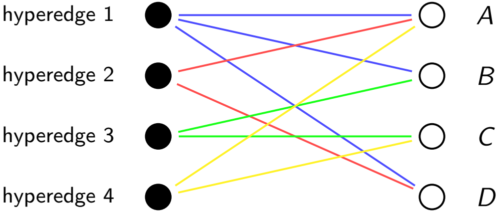
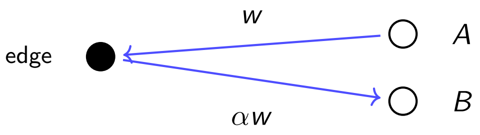




Second generalization: convex sets

- Linear input-output relationship $\longrightarrow$ convex set of *allowable flows* T



Input-output relationship: $h(w) = \alpha w$

$$\text{edge flow} \quad z = \begin{bmatrix} -w \\ h(w) \end{bmatrix}$$

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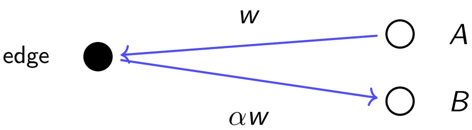

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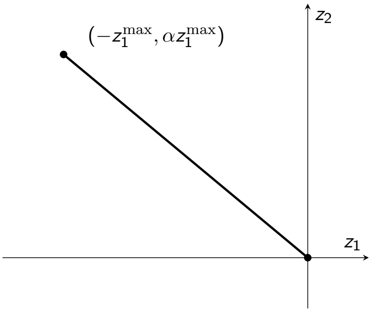

$$T = \{z \in \mathbb{R}^2 \mid z_2 = \alpha \cdot (-z_1)\}$$

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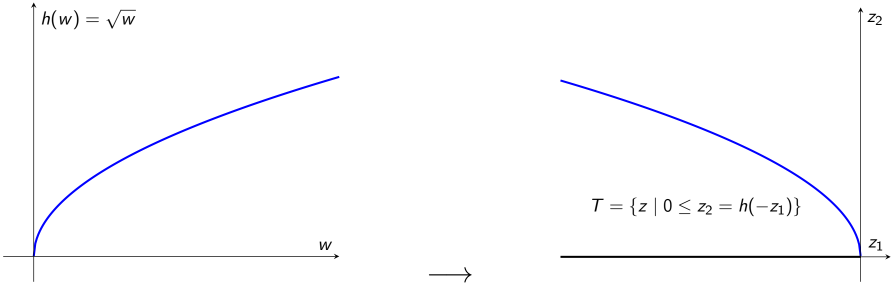

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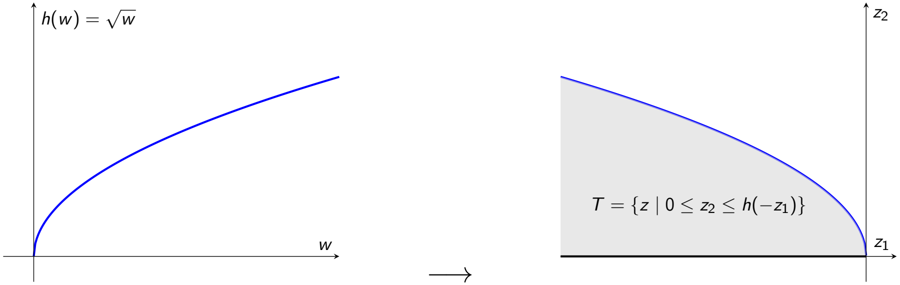

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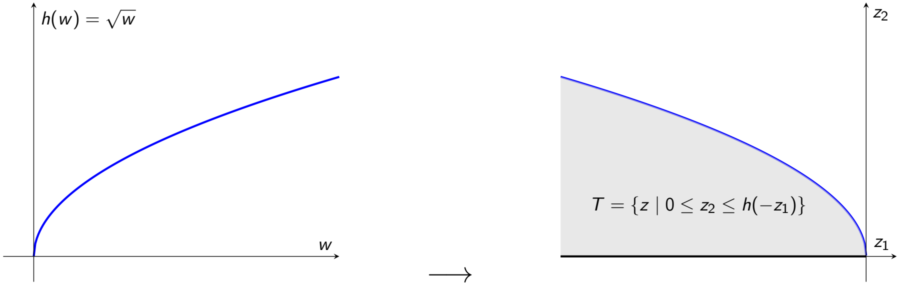

- Claim: we can ensure a solution flow always lies on the boundary (more soon)

Accounting

- Net flow $y \in \mathbb{R}^n$ and edge flows $x_i \in T_i \subseteq \mathbb{R}^{n_i}$

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- Matrices A_i map flow's *local* index in edge i to *global* index, *e.g.*,

node	Local Index	Global Index
B	1	2
D	2	4

$$A_i \cdot \begin{bmatrix} -0.5 \\ 0.3 \end{bmatrix} = \begin{bmatrix} 0 \\ -0.5 \\ 0 \\ 0.3 \\ \vdots \end{bmatrix} \begin{matrix} A \\ B \\ C \\ D \\ \end{matrix}$$

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- The overall *net flow* in the network is

$$y = \sum_{i=1}^{m} A_i x_i$$

Objective: maximize utility

- Concave, increasing utility functions for net flow $U(y)$ and edge flows $\{V_i(x_i)\}$

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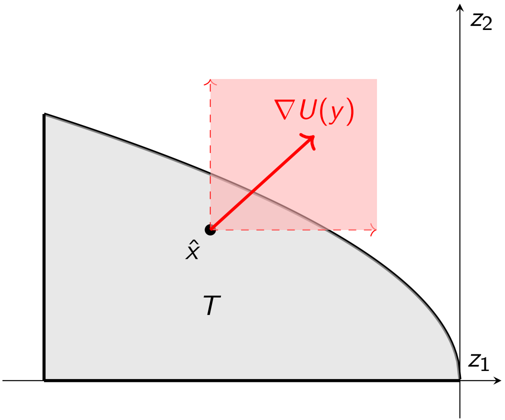

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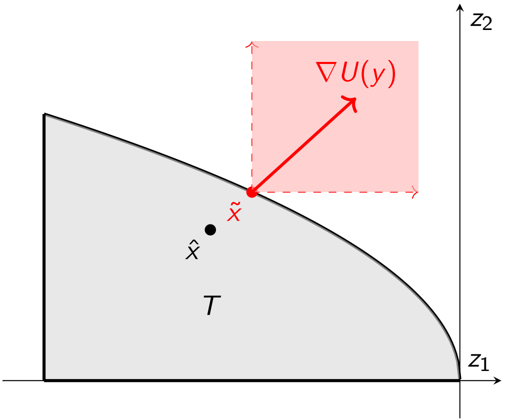

Framework: convex flow problem

(Diamandis et al., arXiv preprint 2024)

- The *convex flow problem*:

$$
\begin{array}{ll}
\text{maximize} & U(y) + \sum_{i=1}^{m} V_i(x_i) \\
\text{subject to} & y = \sum_{i=1}^{m} A_i x_i \\
& x_i \in T_i, \quad i = 1, \ldots, m.
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- **Aside:** definition of allowable flows T_i's allows for a DCP-like 'calculus' of flows

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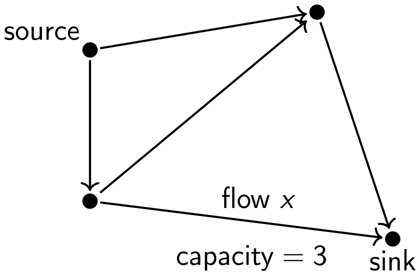
Maximum flow & friends

- Maximum flow problem



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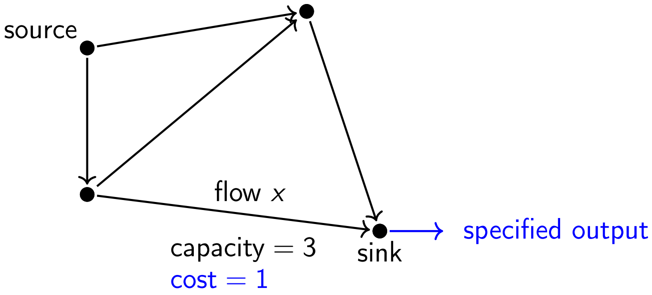
- Minimum cost flow



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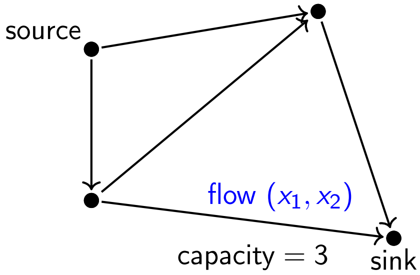

Maximum flow & friends

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- and generalizations...

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(from Stursberg 2019)

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- **Graph:** transmission line network

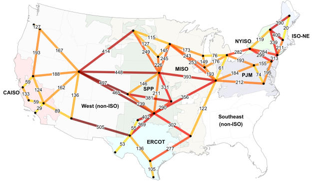

Image source: Lawrence Berkeley National Laboratory

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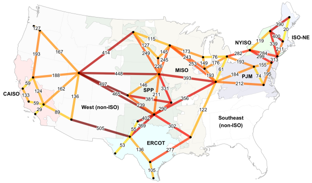

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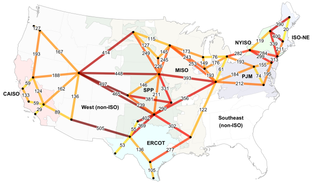

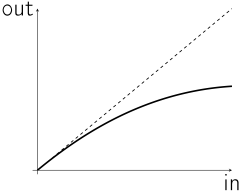

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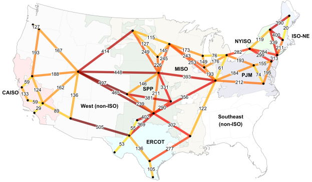

- **Graph:** transmission line network
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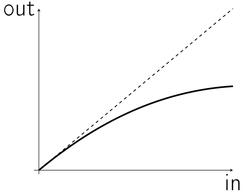
- **Objective:** minimize cost of power generation



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(from Xiao et al. 2004)

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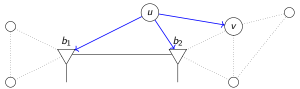

Hyperedges: all outgoing neighbors

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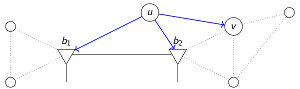

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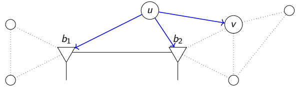

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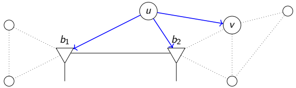

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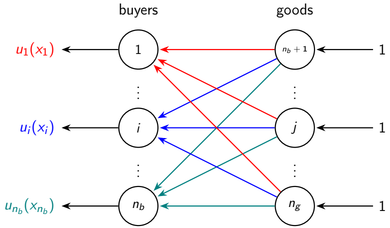

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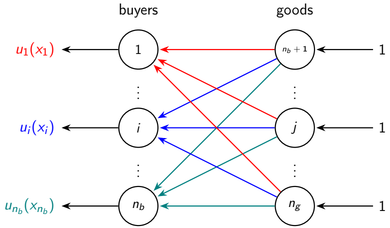

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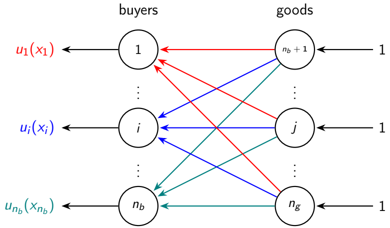

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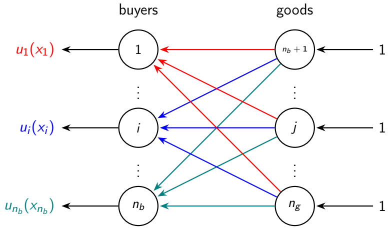
- **Objective:** maximize sum of log utilities



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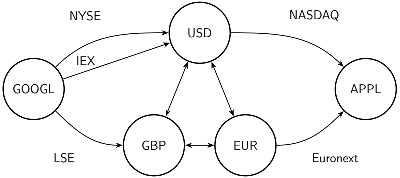

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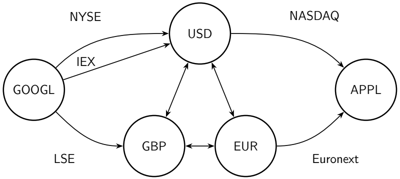

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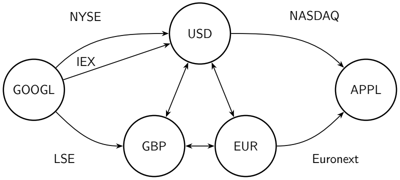

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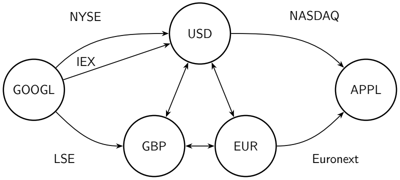

And a lot more...

- Queueing networks (Bertsekas et al. 1992, §5.4)
- Routing games (Roughgarden 2007, §18)
- Supply chains with spoilage (Nagurney et al. 2022, §2.3)
- Reservoir network management (Bertsekas 1998, §8.1)
- Allocating computing resources (Agrawal et al. 2022)
- Supply chain allocation problems (Schütz et al. 2009)
- Wireless network resource allocation (Chiang et al. 2007)

Outline

Motivation

Framework

Applications

Algorithm

Dual problem

- Dual problem:

$$\text{minimize} \quad g(\nu, \eta) = \bar{U}(\nu) + \sum_{i=1}^{m} \left(\bar{V}_i(\eta_i - A_i^T \nu) + f_i(\eta_i) \right),$$

where

$$\bar{U}(\nu) = \sup_y (U(y) - \nu^T y),$$
$$\bar{V}_i(\xi) = \sup_{x_i} (V_i(x_i) - \xi^T x_i),$$
$$f_i(\tilde{\eta}) = \sup_{\tilde{x}_i \in T_i} \tilde{\eta}^T \tilde{x}_i.$$

Subproblems

- Maximum utility problem:

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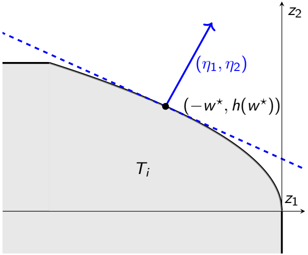

Dual variables as prices

- Optimality conditions:

$$\nabla U(y^\star) = \nu^\star, \qquad \text{`global' net flow prices}$$

$$\nabla V_i(x_i^\star) = \eta_i^\star - A_i^T \nu^\star, \quad i = 1, \ldots, m \qquad \text{`local' edge flow prices}$$

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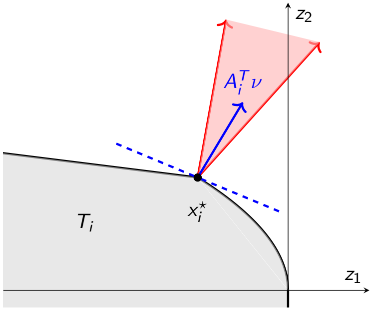

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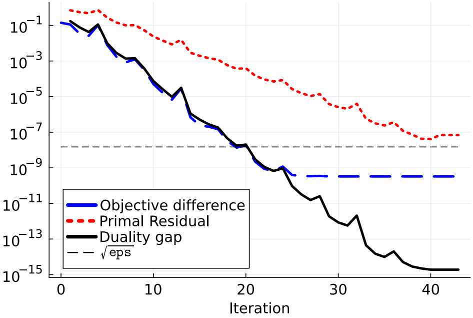
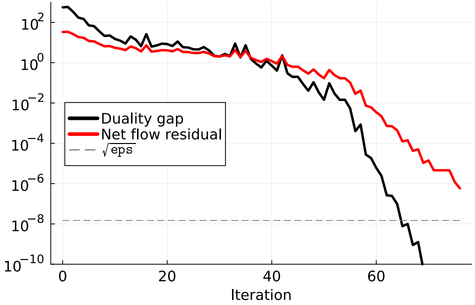
$$

- Can solve dual problem with any first-order method
- If not strictly convex, simple method to recover feasible solution

Example: optimal power flow

- Convergence with L-BFGS-B, after change of variables, (left) and BFGS (right)





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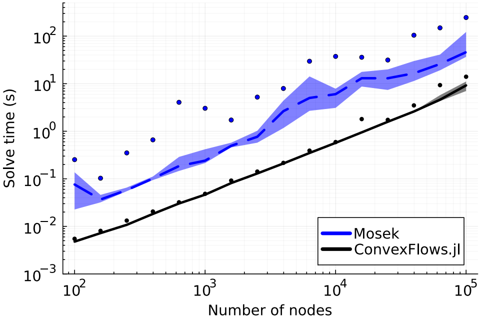

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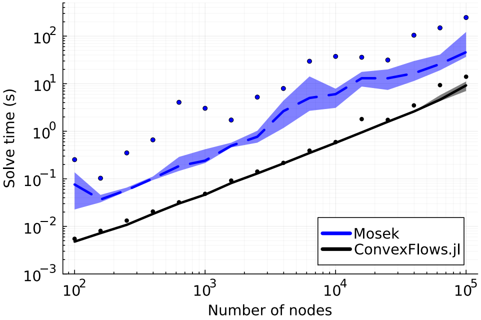
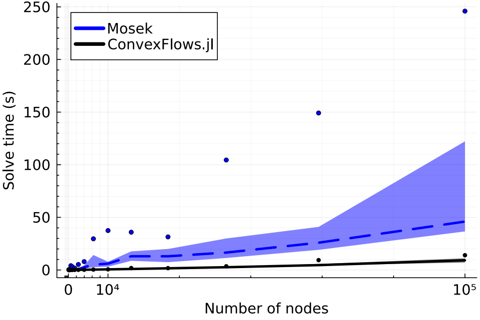
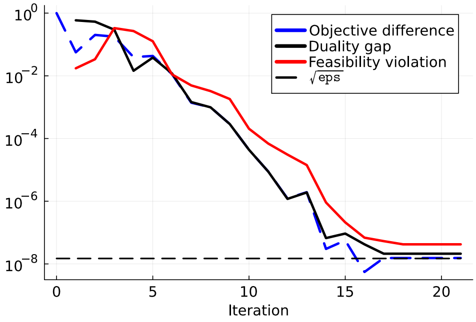
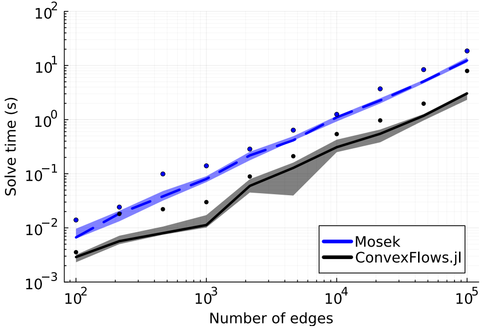




Example: financial order routing





Conjugates? Support functions?

Conjugates? Support functions?

How does a ‘normal’ user specify a problem?

A Simple Interface: `ConvexFlows.jl`

(from Diamandis et al. 2024a)

- Problem specification: library of objective functions, specify edge gain functions:

```
h(w) = 3w - 16.0*(log(1 + exp(0.25 * w)) - log(2))
push!(edges, Edge((i, j); h=h, ub=3.0))
```

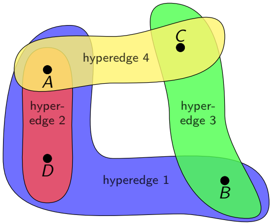
- CVX-like ability to naturally specify problems and quickly test them
- Specify subproblems using `JuMP.jl`
- Performance-sensitive users: specify subproblem solutions directly

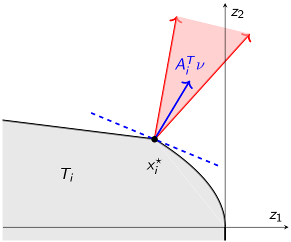
Full problem specification

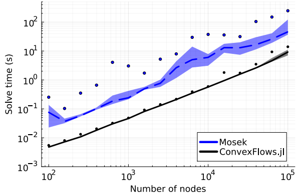
```
# Parameters: demand d, graph Adj, upper bounds ub

obj = NonpositiveQuadratic(d)

h(w) = 3w - 16.0*(log1pexp(0.25 * w) - log(2))

lines = Edge[]
for i in 1:n, j in i+1:n
    Adj[i, j] ≤ 0 && continue
    push!(lines, Edge((i, j); h=h, ub=ub[i]))
end

prob = problem(obj=obj, edges=lines)
result = solve!(prob)
```

Questions? (email: tdiamand@mit.edu)

- New framework for **nonlinear** network flows over **hypergraphs**
  - Generalizes many classic results & includes many problems in the literature
- Fast algorithm to solve, implemented in `ConvexFlows.jl`
- Natural fixed fee and decentralized extensions to model and algorithm







Appendix

Appendix

A bit of history

More theory
- Flow calculus
- Nonconvex flows
- Theory Miscellanea

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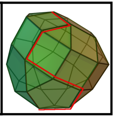

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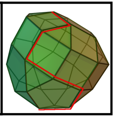

Max flow & min cut
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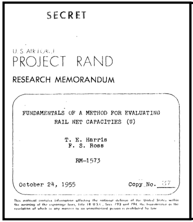
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U.S. AIR FORCE
PROJECT RAND
RESEARCH MEMORANDUM

FUNDAMENTALS OF A METHOD FOR EVALUATING RAIL NET CAPACITIES (U)

T. E. Harris
F. S. Ross

RM-1573

October 24, 1955

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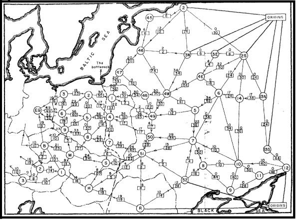

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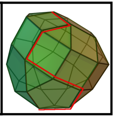

MAXIMAL FLOW THROUGH A NETWORK

L. R. FORD, Jr. and D. R. FULKERSON

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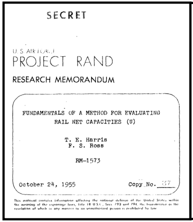

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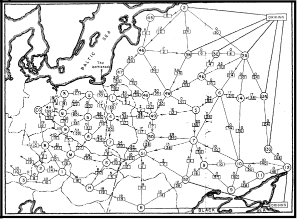

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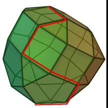

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many applications
today

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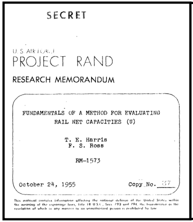
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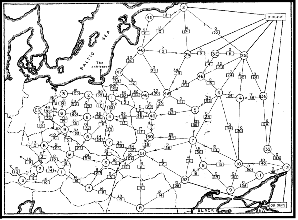

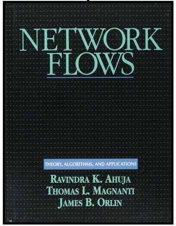

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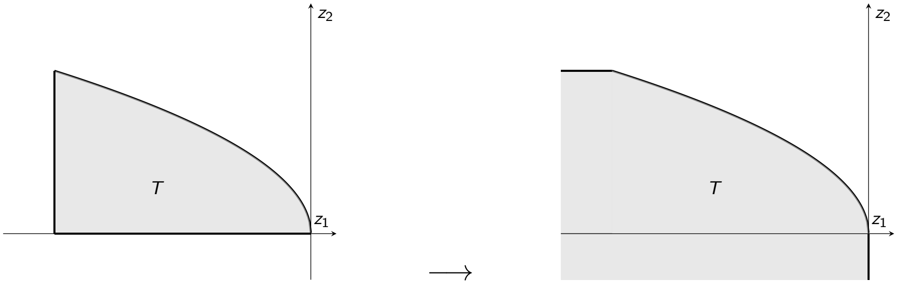
Downward closure

- Idea: allow any 'worse' point



Sets of allowable flows

(inspired by Angeris et al. 2023)

- A set of *allowable flows* $T \subseteq \mathbb{R}^n$ must
  - be closed and convex
  - be downward closed
  - contain the zero vector

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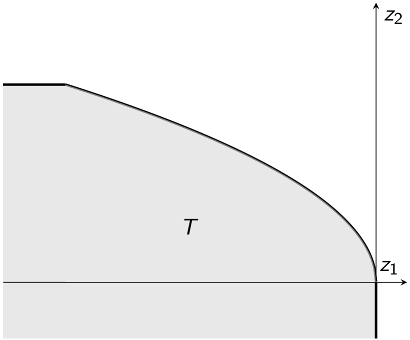
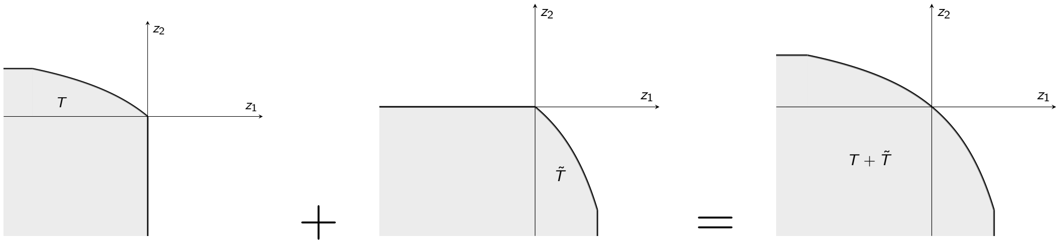


Flow calculus

We can combine, split, and transform sets of allowable flows:

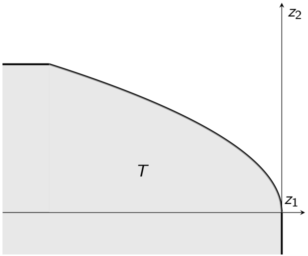
- Addition (Minkowski)



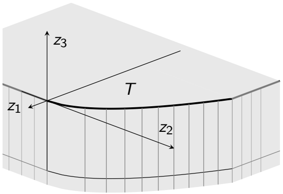
Flow calculus

We can combine, split, and transform sets of allowable flows:

- Addition (Minkowski)
- Scaling by a nonnegative injective matrix: $T_i \longrightarrow AT_i - \mathbb{R}^n$ (*e.g.*, lifting)

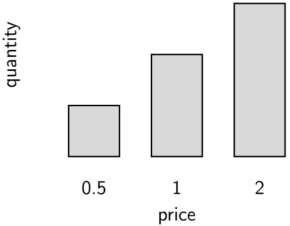


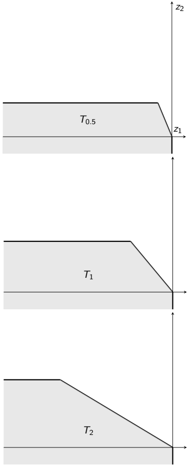
$$\begin{bmatrix} 1 & \\ & 1 \\ & \end{bmatrix} - \mathbb{R}^3 \longrightarrow$$

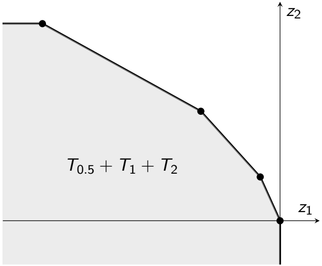


Aggregate edge example

- Orderbook markets: many linear edges or one piecewise linear edge







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Fixed costs

(from Diamandis et al. 2024b)

- Natural extension: **fixed cost** q_i to use edge i

$$
\begin{array}{ll}
\text{maximize} & U(y) + \sum_{i=1}^m V_i(x_i) + q^T \lambda \\
\text{subject to} & y = \sum_{i=1}^m A_i x_i \\
& (x_i, \lambda_i) \in \{(0,0)\} \cup (T_i \times \{-1\}), \quad i = 1, \dots, m,
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- Problem is nonconvex
- NP-hard to solve

Idea: convex relaxation

$$\begin{array}{ll} \text{maximize} & U(y) + \sum_{i=1}^m V_i(x_i) + q^T \lambda \\ \text{subject to} & y = \sum_{i=1}^m A_i x_i \\ & (x_i, \lambda_i) \in \textbf{conv}\left(\{(0,0)\} \cup \left(T_i \times \{-1\}\right)\right), \quad i = 1, \ldots, m, \end{array}$$

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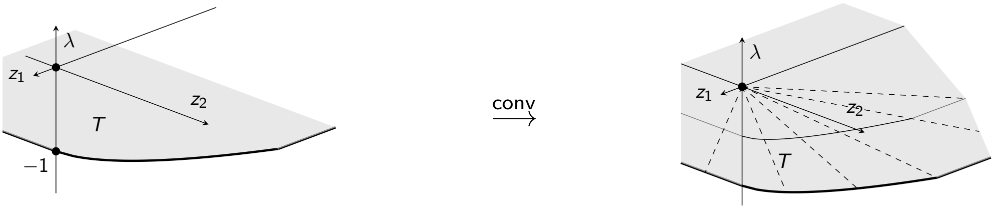

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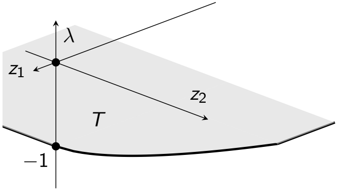
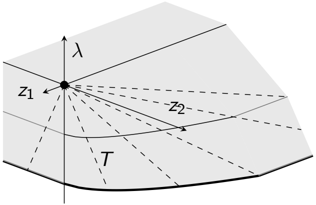


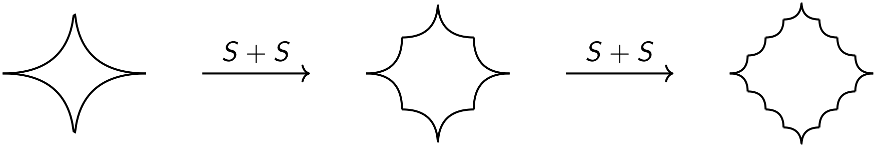
$\xrightarrow{\textbf{conv}}$



$$\textbf{conv}\left(\{(0,0)\} \cup (T_i \times \{-1\})\right) = \textbf{cone}\left(\{(0,0)\} \cup (T_i \times \{-1\})\right) \cap (\mathbb{R}^{n_i} \times [-1, 0]).$$

So what? Case with $V_i = 0$

- Often many more edges m than nodes n
- Shapley–Folkman Lemma: sum of many nonconvex sets is 'almost' convex



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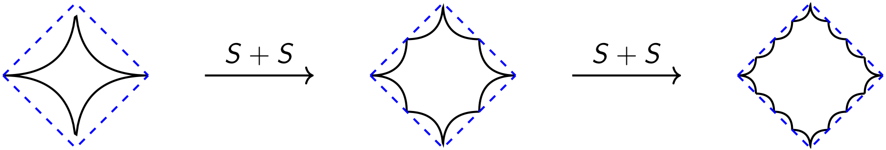

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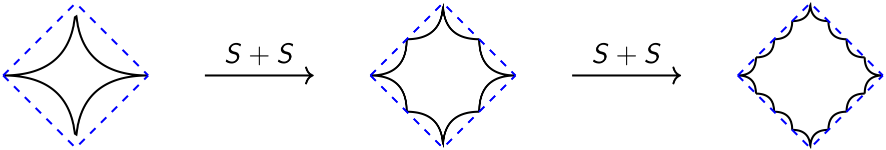

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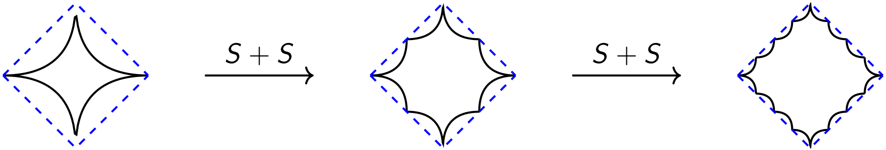

- Can find optimal $\{(x_i, \lambda_i)\}$ with at most $n+1$ non-integral λ_i's
- Optimal objective for relaxation p^0 satisfies $0 \le p^0 - p^\star \le (n+1)\,(\max_i q_i)$.

Structure in the dual problem suggests heuristic

- Recall that the dual problem requires solving the 'arbitrage problem', now

$$f_i^{\mathrm{fees}}(\eta_i) = \max\left(f_i(\eta_i) - q_i,\, 0\right)$$

- Solution $\lambda_i^\star$ is 0 or -1 unless $f_i(\eta_i) = q_i$
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- But may be difficult to find feasible point y if constraints in U.
- Future work: conditions on when heuristic is exact

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An augmenting path generalization

- Max Flow Problem: Flow is optimal iff there is no augmenting path
- Convex Flow Problem ($V_i = 0$): Flow is optimal iff there is no arbitrage

An augmenting path generalization

- Max Flow Problem: Flow is optimal iff there is no augmenting path
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- Define

$$T_i^\star(x_i) = \{\delta \mid x_i + t\delta_i \in T_i \text{ for some } t > 0\}.$$

- No arbitrage condition: for $\nu = \nabla U(y)$ and any $\delta_i \in T_i^\star(x_i)$,

$$\nu^T \left(\sum_{i=1}^{m} A_i \delta_i \right) \leq 0.$$

NP-hard proof

- Knapsack problem: given $c \in \mathbb{Z}_+^n$ and $b \in \mathbb{Z}_+$, find $x \in \{0,1\}^n$ such that $c^T x = b$.
- Reduction of knapsack (subset sum) problem to convex flow problem with fees:

$$\begin{array}{ll} \text{maximize} & y - I(y \geq b) + c^T \lambda \\ \text{subject to} & y = \displaystyle\sum_{i=1}^{m} A_i x_i \\ & (x_i, \lambda_i) \in \{(0,0)\} \cup ((-\infty, c_i] \times \{-1\}), \quad i = 1, \ldots, m. \end{array}$$

- Opt value 0 iff there exists solution to knapsack since

$$y + c^T \lambda = \textstyle\sum_{i=1}^m (-\lambda_i) x_i + c^T \lambda \leq \sum_{i=1}^m c_i \lambda_i (-1 + 1) = 0$$

(Almost) self-dual problem

- Conic problem:

$$\begin{array}{ll} \text{maximize} & U(y) + \sum_{i=1}^m V_i(x_i) \\ \text{subject to} & y = \sum_{i=1}^m A_i x_i \\ & x_i \in K_i, \quad i = 1, \ldots, m. \end{array}$$

- Dual problem:

$$\begin{array}{ll} \text{minimize} & \bar{U}(\nu) + \sum_{i=1}^m \bar{V}_i(\eta_i - \xi_i) \\ \text{subject to} & \nu = \sum_{i=1}^m A_i \xi_i \\ & \eta_i \in K_i^\circ, \quad i = 1, \ldots, m. \end{array}$$

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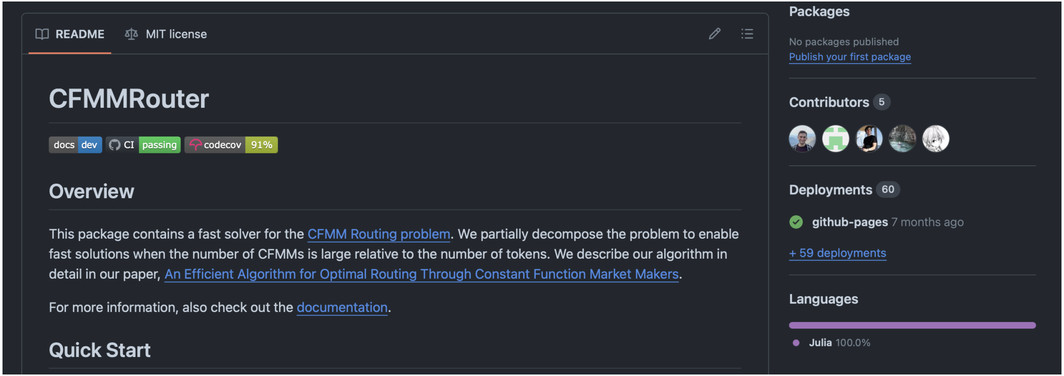
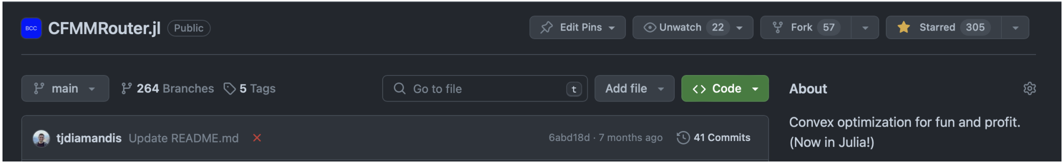
My 2022 internship project



It became quite popular...



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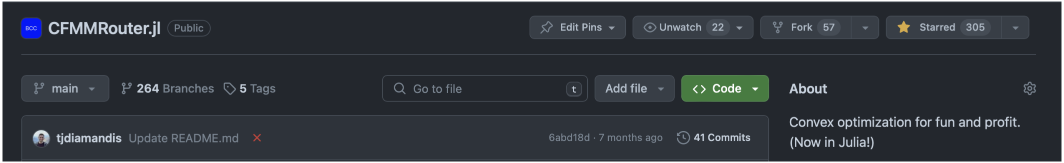
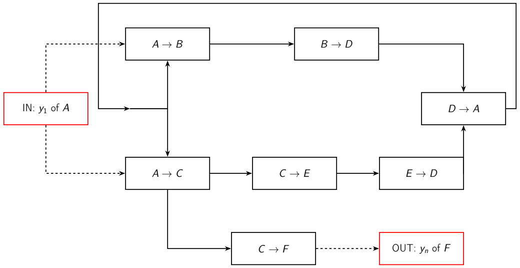


- A company (Flood) reached out to implement this (exact same algorithm)
- And I decided to write up a paper on it for Financial Cryptography

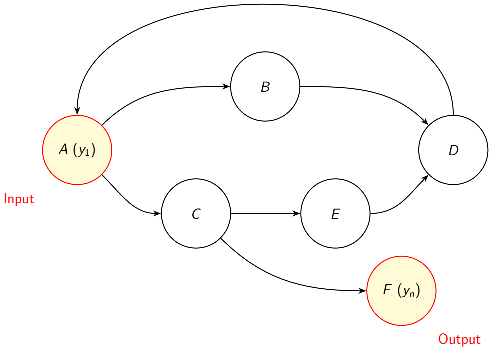
A real example

- Trade asset A for asset F

Markets as nodes:



Assets as nodes:



A real example

- This one was executed in reality by Flood (`flood.bid`)

From	0x41850fe2843b8...	To	0xf111ed85e40931...	For	23.760766	($23.86)	USD Coin (USDC)
From	DODO: USDT-US...	To	0xf111ed85e40931...	For	23.761929	($23.81)	Tether USD (USDT)
From	0xf111ed85e40931...	To	DODO: USDT-US...	For	23.780712	($23.88)	Bridged USDC (USDC.e)
From	DODO: USDT-US...	To	DODO: Multisig W...	For	0.000237	($0.00)	Tether USD (USDT)
From	0x7fcdc35463e37...	To	0xf111ed85e40931...	For	0.005066974652006983	($16.79)	Wrapped Ethe... (WETH)
From	0xf111ed85e40931...	To	0x7fcdc35463e37...	For	17.162688	($17.23)	USD Coin (USDC)
From	0xf111ed85e40931...	To	0xc082398767ae7...	For	2.817007	($2.83)	Bridged USDC (USDC.e)
From	0xc082398767ae7...	To	0xf111ed85e40931...	For	0.00004204	($2.78)	Wrapped BTC (WBTC)
From	0x562d29b54d2c5...	To	0xf111ed85e40931...	For	26.598204	($26.70)	Bridged USDC (USDC.e)
From	0xf111ed85e40931...	To	0x562d29b54d2c5...	For	26.597903	($26.70)	USD Coin (USDC)
From	0xf111ed85e40931...	To	0x7050a8908e2a6...	For	0.005066974652006984	($16.79)	Wrapped Ethe... (WETH)
From	0x7050a8908e2a6...	To	0xf111ed85e40931...	For	0.953228882631324394	($17.17)	ChainLink To... (LINK)
From	0x655c1607f8c2e...	To	0xf111ed85e40931...	For	20.000907	($20.08)	USD Coin (USDC)
From	0xf111ed85e40931...	To	0x655c1607f8c2e...	For	1.109994504624138108	($19.99)	ChainLink To... (LINK)
From	0xa79fd76ca2b24...	To	0xf111ed85e40931...	For	0.156826706956773007	($2.82)	ChainLink To... (LINK)
From	0xf111ed85e40931...	To	0xa79fd76ca2b24...	For	0.00004204	($2.78)	Wrapped BTC (WBTC)
From	0xf111ed85e40931...	To	0x41850fe2843b8...	For	23.761929	($23.81)	Tether USD (USDT)

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Optimal power flow

- Goal: minimize a quadratic power generation cost for random demand

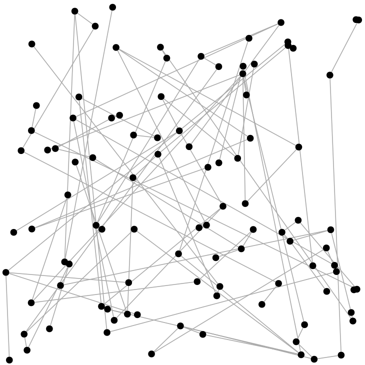
$$c_i(w) = \begin{cases} (1/2)w^2 & w \geq 0 \\ 0 & w < 0. \end{cases}$$

- Demand randomly sampled from $\{0.5, 1, 2\}$
- Power each node needs to generate is $d - y$, so objective is

$$U(y) = \sum_{i=1}^{n} -c_i(d_i - y_i).$$

Optimal power flow: network

- We generate the network as in Kraning et al. 2013



Optimal power flow: edges

- Edges have random capacity, sampled from $\{1, 2, 3\}$
- Power lost is
$$\ell(w) = 16\left(\log(1 + \exp(w/4)) - \log 2\right) - 2w$$
- The set of allowable flows is
$$T = \{z \in \mathbb{R}^2 \mid -b \leq z_1 \leq 0 \ z_2 \leq -z_1 - \ell(-z_1)\}$$
- Given prices η, the optimal input has a closed form:
$$x_1^\star = -\left(4\log\left(\frac{3\eta_2 - \eta_1}{\eta_2 + \eta_1}\right)\right)_{[0,b]}, \qquad x_2^\star = -x_1^\star - \ell(-x_1^\star)$$

Optimal power flow: conic form

- Compare the convex flow problem with the equivalent conic form

$$
\begin{array}{ll}
\text{maximize} & -1^T t_1 \\
\text{subject to} & (0.5,\, (t_1)_i,\, (t_2)_i) \in K_{\mathrm{rot2}}, \quad \text{for } i = 1, \dots n \\
& t_2 \geq d - y, \qquad t_2 \geq 0 \\
& -b_i \leq (x_i)_1 \leq 0, \quad \text{for } i = 1, \dots m \\
& u_i + v_i \leq 1 \quad \text{for } i = 1, \dots m \\
& (-\beta_i (x_i)_1 + (3(x_i)_1 + (x_i)_2)/\alpha - \log(2),\, 1,\, u_i) \in K_{\mathrm{exp}} \quad \text{for } i = 1, \dots m \\
& ((3(x_i)_1 + (x_i)_2)/\alpha - \log(2),\, 1,\, v_i) \in K_{\mathrm{exp}} \quad \text{for } i = 1, \dots m.
\end{array}
$$

Financial network routing problem: edges

- Most DEXs are implemented as *constant function market makers* (CFMMs)
- CFMMs are defined by their trading function $\varphi : \mathbb{R}_+^n \to \mathbb{R}$
- Maps reserves $R \in \mathbb{R}_+^n$ to a real number
- Is concave and increasing
- Accepts trade $\Delta \to \Lambda$ if $\varphi(R + \gamma\Delta - \Lambda) \geq \varphi(R)$.

Financial network routing problem: conic form

$$
\begin{array}{ll}
\text{maximize} & c^T y - (1/2)\sum_{i=1}^{n}(p_1)_i - (1/2)\sum_{i=1}^{m}(t_1)_i \\
\text{subject to} & (0.5,\, (p_1)_i,\, (p_2)_i) \in K_{\mathrm{rot2}}, \quad i = 1, \ldots, n \\
& p_1 \geq 0 \\
& p_2 \geq 0, \quad p_2 \geq -y \\
& (0.5,\, (t_1)_i,\, (t_2)_i) \in K_{\mathrm{rot2}}, \quad i = 1, \ldots, n \\
& t_1 \geq 0 \\
& t_2 \geq 0, \quad (t_2)_i \geq -(\Lambda_i - \Delta_i) \\
& (R + \gamma\Delta - \Lambda,\, \varphi(R)) \in K_{\mathrm{pow}}(w_i), \quad i = 1, \ldots, m_1 \\
& (-3\varphi(R),\, R + \gamma\Delta - \Lambda) \in K_{\mathrm{geomean}}, \quad i = m_1 + 1, \ldots, m \\
& \Delta_i,\, \Lambda_i \geq 0, \quad i = 1, \ldots, m,
\end{array}
$$

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Full dual problem

- $\bar{U}$ and $\bar{V}_i$ introduce implicit nonnegativity constraints
- Dual problem with these explicit is:

$$\begin{array}{ll} \text{minimize} & \bar{U}(\nu) + \sum_{i=1}^{m} \Big(\bar{V}_i(\eta_i - A_i^T \nu) + f_i(\eta_i) \Big) \\ \text{subject to} & \nu \geq 0, \quad \eta_i \geq A_i^T \nu, \quad i = 1, \ldots, m. \end{array}$$

- Letting $\mu = (\nu, \eta)$, a change of variables gives

$$\begin{array}{ll} \text{minimize} & g(F^{-1}\tilde{\mu}) \\ \text{subject to} & \tilde{\mu} \geq 0, \end{array}$$

Two-node subproblems

- The arbitrage problem for two nodes is

$$f(\eta) = -\eta_1 w + \eta_2 h(w)$$

- This has optimality conditions

$$\eta_2 h^+(w^\star) \leq \eta_1 \leq \eta_2 h^-(w^\star)$$

- When differentiable, forward and reverse derivatives equal

Second-stage problem

- Assume U strictly concave (so $y^\star$ unique)
- Let S be the set of strictly concave allowable flows
- Second-stage problem:

$$\begin{array}{ll} \text{minimize} & \|y^\star - \sum_{i=1}^{m} A_i x_i\| \\ \text{subject to} & x_i = \tilde{x}_i^\star, \qquad\qquad i \in S \\ & x_i \in T_i \cup \partial f_i(\eta_i^\star), \quad i \notin S. \end{array}$$

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